A HECKE CORRESPONDENCE THEOREM FOR NONANALYTIC AUTOMORPHIC INTEGRALS

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Abstract. In this paper we prove a Riemann-Hecke-Bochner correspondence for nonanalytic automorphic integrals on the Hecke groups. We also present several applications of this theorem. One of these settles a question, posed by M. Knopp in 1983, regarding the Mellin transform of modular integrals .

1. Introduction

In this work we continue the line of inquiry which commenced with [**26**]. In that earlier paper, we discussed (but did not prove) a Riemann-Hecke-Bochner correspondence theorem for *nonanalytic automorphic integrals*, that is, functions of the form

$$f(z)=\sum_{m=1}^{M} y^{w_m} \sum_{n_1,n_2=0}^{\infty} a_{n_1,n_2,m} \exp\left\{\frac{2\pi i}{\lambda}\left(n_1 z-n_2\overline{z}\right)\right\},$$

$y=\mathrm{Im} z>0$, which satisfy a transformation law:

$$z^{-\alpha}\left(\overline{z}\right)^{-\beta} f(-1/z)=Cf(z)+q(z);$$

here w_m, $a_{n_1,n_2,m}$, α, β, C are complex numbers and λ is a positive real, with the $a_{n_1,n_2,m}$ adhering to a further technical requirement. (To make this transformation law meaningful, of course, q needs to be described more explicitly. We will assume that q is an *axial log-polynomial sum*; that is,

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$$q(iy) = \sum_{j=1}^{J} (iy)^{\alpha_j} \sum_{t=0}^{T} \beta_{j,t} [\log(iy)]^t \text{ for } y > 0,$$

with α_j and $\beta_{j,t}$ complex.)

Our motivation in defining integrals thusly is threefold. First, this space of quasi-invariant functions is preserved under the weight-changing operators outlined in [**22**, **5**, **14**, **26**]. Moreover, it includes all of the familar examples of *entire* automorphic integrals (e.g., Hurwitz's weight 2 Eisenstein series, the classical theta and eta functions, and so forth), and thereby is indeed a generalization of the more familiar concept. Finally, as we will show, it is possible to state a Hecke correspondence for such functions in a natural way.

Our goal is to demonstrate a correspondence which matches each nonanalytic automorphic integral with a linear combination of Dirichlet series, said linear combination satisfying a functional equation similar to that of the Riemann zeta function. Along the way we will acquire several results which are immediate consequences of the main theorem.

2. Definitions

The *Hecke group* is $\mathcal{G}_\lambda = \left\langle \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$, $\lambda > 0$. For brevity, we refer to these two generators as S_λ and T respectively. The elements of $\mathcal{G}_\lambda$ act on $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ as linear fractional transformations. Of particular interest is the *modular* group $\mathcal{G}_1$.

For $z, w \in \mathbb{C}$, $z \neq 0$, we define $z^w = e^{w \log z}$. Here $\log z = \log|z| + i \arg z$, with $\log|z|$ denoting the principal branch of the logarithm ($\log 1 = 0$); $\arg z$ is taken in the interval $[-\pi, \pi)$, except when a 'binary' convention is more convenient (it will always be clear from context which of these is intended). A *multiplier system on*

$\mathcal{G}_\lambda$ *of coweights* $\alpha, \beta \in \mathbb{C}$ is a function $\upsilon : \mathcal{G}_\lambda \to \mathbb{C}$ satisfying $|\upsilon(S_\lambda)| = 1$, $\upsilon(T) \neq 0$ and the *consistency condition:*

$$\upsilon(M_3)(c_3 z + d_3)^\alpha (c_3 \bar{z} + d_3)^\beta =$$

$$\upsilon(M_1)(c_1 M_2 z + d_1)^\alpha (c_1 M_2 \bar{z} + d_1)^\beta \upsilon(M_2)(c_2 z + d_2)^\alpha (c_2 \bar{z} + d_2)^\beta,$$

for all $M_1, M_2 \in \mathcal{G}_\lambda$, $M_1 M_2 = M_3$, $M_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ for $j = 1, 2, 3$, $z \in \mathcal{H}$, where we interpret the consistency condition according to the *binary argument convention*: $-\pi \leq \arg(cz + d) < \pi$ and $-\pi < \arg(c\overline{z} + d) \leq \pi$, for $z \in \mathcal{H}$, $|c| + |d| \neq 0$. (See, for example, [**22**].)

In this work we consider only multiplier systems which satisfy $\upsilon(S_\lambda) = 1$. More general multiplier systems are examined in [**25**].

Definition 2.1. *Let* $\{a_n\}_{n=0}^\infty \subseteq \mathbb{C}$ *such that* $a_n = O(n^\gamma)$ *as* $n \to \infty$, *for some* $\gamma \in \mathbb{R}^+$. *Write*

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda},$$

$z \in \mathcal{H}$. *Let* υ *be a multiplier system on* $\mathcal{G}_\lambda$ *of real coweights* $k, 0$ *with* $\upsilon(S_\lambda) = 1$. *If* $z^{-k} f(-1/z) = \upsilon(T) f(z) + q(z)$ *for all* $z \in \mathcal{H}$, *where*

$$q(z) = \sum_{j=1}^{J} z^{\alpha_j} \sum_{t=0}^{T} \beta_{j,t} (\log z)^t,$$

$\alpha_j, \beta_{j,t} \in \mathbb{C}$, *we call* f *an* ***automorphic integral*** *of coweights* α, β *and multiplier system* υ *on* $\mathcal{G}_\lambda$. *The function* $q(z)$ *is called a* ***log-polynomial sum*** *(more specifically,* the *log-polynomial period function for* f*).*

Remark on terminology. Some authors refer to these as *entire*, to distinguish them from general automorphic integrals, which may have poles in $\mathcal{H}$ or at the infinite cusp.

The log-polynomial sums which occur as period functions for (entire) automorphic integrals of coweights $k, 0$ have been completely characterized in the cases $k > 2$, $\upsilon(S_\lambda) = 1$ and $k > 0$, $\upsilon(S_\lambda) \neq 1$ [**8**, **9**].

Definition 2.2. *Let* $\{a_{n_1,n_2,m} \mid 0 \leq n_1, n_2 < \infty,\ 1 \leq m \leq M\} \subseteq \mathbb{C}$, *with*

$$\sum_{n_1+n_2=n} a_{n_1,n_2,m} = O\left(n^\gamma\right),\ \gamma > 0,\ \text{as } n \to +\infty.$$

Put

$$f(z) = \sum_{m=1}^{M} \sum_{n_1,n_2=0}^{\infty} y^{w_m} a_{n_1,n_2,m} e^{\frac{2\pi i}{\lambda}(n_1 z - n_2 \overline{z})},$$

$z = x + iy \in \mathcal{H}$. $(w_1, ...w_M \in \mathbb{C}.)$

Let υ *be a multiplier system on* G_λ *of complex coweights* α, β *with* $\upsilon(S_\lambda) = 1$. *If* f *satisfies*

$$z^{-\alpha}\left(\overline{z}\right)^{-\beta} f(-1/z) = \upsilon(T) f(z) + q(z)$$

for all $z \in \mathcal{H}$, *where*

$$q(iy) = \sum_{j=1}^{J} (iy)^{\alpha_j} \sum_{t=0}^{T} \beta_{j,t} \left[\log(iy)\right]^t,\ y > 0,$$

we call f *a* ***nonanalytic automorphic integral*** *of coweights* α, β *and multiplier system* υ *on* $\mathcal{G}_\lambda$. *(*q *is the axial log-polynomial period function for* f.*)*

If one writes f as a function of $u = e^{2\pi i z/\lambda}$, one obtains the "q-like" series [**24**]:

$$g(u) = \sum_{m=1}^{M} \sum_{n_1,n_2=0}^{\infty} b_{n_1,n_2,m} u^{n_1} \overline{u}^{n_2} \log^{w_m} |u|,\ 0 < |u| < 1.$$

A restricted case of the nonanalytic automorphic integral was considered by Knopp [**18**]: namely, the case $\alpha = -\beta \in \mathbb{Z}$, $\{\omega_m\} \subseteq \mathbb{Z}$, and $a_{n_1,n_2,m} = 0$ for $|n_1| + |n_2| > 0$. Also, not all axial log-polynomial periods were considered there, but only those of the form

$$q(z) = \sum_{\text{finite}} \left[\alpha_1 z^{\beta_1} \overline{z}^{\beta_2} (\log z)^t + \alpha_2 z^{\gamma_1} \overline{z}^{\gamma_2} (\log \overline{z})^u\right]$$

In particular, Knopp developed a direct Hecke theorem for those integrals.

A remark regarding the last two definitions: If $\lambda = 1$, f is said to be *modular*; if $q \equiv 0,$ the (analytic or nonanalytic) *integral* is called a *form*.

In certain contexts one may replace the (axial) log-polynomial sum by an (axial) rational function. Rational period functions have been the subject of a considerable body of recent research [**2, 3, 4, 6, 7, 10, 11, 13, 14, 16, 23**].

Remark 2.1. *Some authors assume seemingly weaker conditions for Definition 2.1, namely that f is entire, periodic, and bounded at $i\infty$, and the rest of the definition follows. Observe that no such set of conditions will replace Definition 2.2; real-analyticity and periodicity do not necessarily imply the quasi-exponential shape we specified, nor does the function need to be bounded at $i\infty$.*

Nevertheless, we feel that this definition is a natural one for the reasons stated in the introduction.

3. Riemann-Hecke-Bochner Correspondence

The Riemann-Hecke-Bochner Correspondence asserts that there is a relationship between exponential series which satisfy a transformation law and Dirichlet series with a certain type of functional equation. Such theorems originate with Riemann's proof [**27**] of the functional equation for his eponymous zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

by way of the classical theta function

$$\vartheta(z) = 1 + 2\sum_{n=1}^{\infty} e^{\pi i n^2 z},$$

a modular form of weight $\frac{1}{2}$ on $\mathcal{G}_2$ [**17**]. ζ and ϑ are connected by the Mellin transform and its inverse; this connection was later generalized to the case of automorphic forms by Hecke [**12**] and still later to automorphic integrals by Bochner [**1**; see also

30], who allowed "residual" period functions. These were later described explicitly as log-polynomial sums by Knopp, who also elaborated on the relationship between the period functions and the poles of the Dirichlet series; Knopp's incarnation of the correspondence follows [**19, 20**]:

Theorem 3.1. *Let* $k \in \mathbb{R}$, $C \in \mathbb{C}$, $\lambda > 0$. *Suppose*

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda}$$

for $z \in \mathcal{H}$, *where* $a_n = O(n^{\gamma})$ *as* $n \to \infty$, *for some* $\gamma > 0$. *Put*

$$\Phi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) \sum_{n=1}^{\infty} a_n n^{-s},$$

Res *large. Then* (a) $\Leftrightarrow$ (b), *where:*

(a) $z^{-k} f(-1/z) = C f(z) + q(z)$ *for all* $z \in \mathcal{H}$, *with* $q(z) = \sum_{j=1}^{J} z^{\alpha_j} \sum_{t=0}^{M_j} \beta_{j,t} (\log z)^t$ $(\alpha_j, \beta_{j,t} \in \mathbb{C})$.

(b) *(i)* Φ *has a meromorphic continuation to* $\mathbb{C}$ *with at most a finite number of poles, and*

(ii) Φ *is bounded in each set of the form*

$$L(\sigma_1, \sigma_2, t_0) = \{s = \sigma + it : \sigma_1 \le \sigma \le \sigma_2,\ |\mathrm{Im} t| \ge t_0\},$$

whenever $\sigma_1, \sigma_2 \in \mathbb{R}$ *and* $t_0 > \max_j |\mathrm{Im}\alpha_j|$ *(figure 1), and*

(iii) Φ *has the functional equation:*

$$\Phi(k - s) = e^{\pi i k/2} \Phi(s).$$

Knopp refers to the set $L(\sigma_1, \sigma_2, t_0)$ *as a* lacunary vertical strip.

Supplement to Theorem 3.1. The locations and orders of the poles of Φ are related to q. In fact, the principal part of Φ at each pole can be written explicitly in terms of the α_j's and $\beta_{j,t}$'s.

Again, one may also consider *rational* q here instead of log-polynomials [**10, 11, 4**].

We omit the proof of Theorem 3.1, since it is a consequence of our next theorem. *Note:* The above condition $t_0 > \max_j |\mathrm{Im}\alpha_j|$ is a slight, but essential, correction to the statement which appears in [**19**].

It is important to observe that the transition from forms to integrals with log-polynomial period functions does not disturb the functional equation of Φ; the only difference is in the placement and orders of the poles of Φ. Likewise, our foray into the nonanalytic arena will leave the functional equation essentially undisturbed; the main difference is that now Φ is a *linear combination* of Dirichlet series with exponential and gamma factors. This result, which follows, is suggested by unpublished work of Knopp ([**18**]; see also Remark 6 in the next section).

Theorem 3.2. *Let* $\lambda, \gamma > 0$; $C, \alpha, \beta \in \mathbb{C}$; *and* $w_m, a_{n_1,n_2,m} \in \mathbb{C}$, w_m *distinct, for* $m = 1, 2, ..., M$ *and* $n_1, n_2 \in \mathbb{Z}^+ \cup \{0\}$. *Define* $c_{n,m} = \sum_{n_1+n_2=n} a_{n_1,n_2,m}$ *for* $m = 1, 2, ..., M$ and $n \in \mathbb{Z}^+ \cup \{0\}$. *Assume also that for each* m, $\sum_{n_1+n_2=n} |a_{n_1,n_2,m}| = O(n^\gamma)$ *as* $n \to \infty$. *For* $z = x + iy \in \mathcal{H}$, *define the real-analytic periodic function*

$$f(z) = \sum_{m=1}^{M} y^{w_m} \sum_{n_1,n_2=0}^{\infty} a_{n_1,n_2,m} \exp\left\{\frac{2\pi i}{\lambda}(n_1 z - n_2 \overline{z})\right\}.$$ *Also define*

$$\Phi_f(s) = \sum_{m=1}^{M} (2\pi/\lambda)^{-s-w_m} \Gamma(s + w_m) \sum_{n=1}^{\infty} c_{n,m} n^{-s-w_m},$$ *for* $\mathrm{Re}\, s$ *large.*

Then the following are equivalent:

($\mathcal{A}$) $z^{-\alpha}(\overline{z})^{-\beta} f(-1/z) = Cf(z) + q(z)$ *for* $z \in \mathcal{H}$, *with* $q(iy) = \sum_{j=1}^{J} (iy)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{j,t} (\log iy)^t$, *for* $y > 0$. *(*$\alpha_j, \beta_{j,t} \in \mathbb{C}$, α_j *distinct,* $\beta_{j,M_j} \neq 0\ \forall j$*).*

($\mathcal{B}$) *(i)* $\Phi_f(s)$ *has meromorphic continuation to* $\mathbb{C}$ *with at most a finite number of poles;*

(ii) $\Phi_f(s)$ *is bounded in sets of the form*

$$L(\sigma_1, \sigma_2, t_0) = \{s = \sigma + it : \sigma_1 \le \sigma \le \sigma_2,\ |\mathrm{Im}\, t| \ge t_0\}$$

whenever $\sigma_1, \sigma_2 \in \mathbb{R}$ *and*

$$t_0 > \max_j |\mathrm{Im}\, \alpha_j| + \max_m |\mathrm{Im}\, w_m| + \max_m |\mathrm{Im}\,(w_m + \alpha + \beta)|\,;$$

(iii) $\Phi_f(\alpha + \beta - s) = i^{\alpha-\beta} C\, \Phi_f(s)$, $s \in \mathbb{C}$.

Corollary (to the proof of Theorem 3.2). When $\mathcal{A}$, $\mathcal{B}$ hold, it is also true that $\Phi_f - r - L$ is entire, where

$$r(s) = \sum_{m=1}^{M} c_{o,m} \left[\frac{i^{\alpha-\beta} C}{s - (\alpha + \beta + w_m)} - \frac{1}{s + w_m} \right]$$

and

$$\begin{aligned} L(s) &= \sum_{j=1}^{J} i^{\alpha-\beta+\alpha_j} \sum_{t=0}^{M_j} \beta_{j,t} \sum_{\ell=0}^{t} \binom{t}{\ell} \left(\frac{i\pi}{2}\right)^{t-\ell} \ell!\, [s - (\alpha + \beta + \alpha_j)]^{-\ell-1} \\ &= -C^{-1} \sum_{j=1}^{J} i^{\alpha_j} \sum_{t=0}^{M_j} \beta_{j,t} \sum_{\ell=0}^{t} (-1)^{\ell} \binom{t}{\ell} \left(\frac{i\pi}{2}\right)^{t-\ell} \ell!\, (s + \alpha_j)^{-\ell-1}. \end{aligned}$$

Thus the location of the poles of Φ_f and their orders are obvious. In particular, we have the set identity $\{\alpha + \beta + \alpha_j\}_{j=1}^{J} = \{-\alpha_j\}_{j=1}^{J}$.

Theorem 3.2 and the corollary are proved in the final section.

4. Remarks

1. Theorem 3.2 still holds even if f is defined only on the positive imaginary axis (as $\sum_m (iy)^{w_m} \sum_n c_{n,m} e^{-2\pi n y/\lambda}$). Thus, periodicity and real-analyticity are unnecessary hypotheses, but we will retain them because we are interested in nonanalytic integrals on $\mathcal{H}$. (For one thing, there is ambiguity regarding the meaning of weight-changing operators such as

$$\delta_{\alpha,\beta} = \frac{\partial}{\partial z} + \frac{\alpha}{2iy},\ \partial_{\alpha,\beta} = y^2 \frac{\partial}{\partial \overline{z}} + \frac{\beta i y}{2},$$

etc. when they are applied to functions only defined on $i\mathbb{R}^+$.)

2. It is tempting to conclude that Theorem 3.2 can be reduced to the analytic case (Theorem 3.1), but while there does exist f_1 analytic on $\mathcal{H}$ with $(f - f_1)\,|_{i\mathbb{R}^+} = 0$, f_1 does not transform correctly under S_λ if $M > 1$. Thus, while f_1 may be useful to us (and we will have cause to refer to just such a "corresponding analytic function" in the proof of Lemma 6.1), it is not an analytic integral in general.

3. Φ_f is the *Mellin transform* of f, or more precisely of $f - \sum_{m=1}^{M} c_{o,m} i^{-w_m} z^{w_m}$. That is,

$$\Phi_f(s) = \int_0^\infty \left[f(iy) - \sum_{m=1}^{M} c_{o,m} i^{-w_m} (iy)^{w_m} \right] y^{s-1} dy,$$

for $\mathrm{Re}\, s$ large.

4. In Theorem 3.2, take $\beta = 0$, $\alpha \in \mathbb{R}$, $M = 1$ and $w_1 = 0$ (so that f, g are analytic) to get Theorem 3.1.

5. In Theorem 3.1, Φ determines f up to the constant term. In Theorem 3.2, however, Φ_f determines only $f(iy)$ uniquely (again, up to the constant term), not $f(z)$. Thus, Theorem 3.2 is a one-to-one correspondence between linear combinations of Dirichlet series satisfying a functional equation, and *equivalence classes* of quasi-exponential functions satisfying a transformation law, where we say two functions are equivalent if their difference vanishes identically on the positive imaginary axis.

6. Theorem 3.2 was inspired by a result presented in [**18**], which is a direct theorem ($\mathcal{A} \Rightarrow \mathcal{B}$) for the case $\beta = -\alpha \in \mathbb{Z}$, $C = 1$, $\{w_m\} \subseteq \mathbb{Z}$ and $a_{n_1,n_2,m}$ supported only when n_1 or $n_2 = 0$. Also, no converse was given there, although it was suggested that one should be found. In addition, the period functions had the form

$$\sum_{\text{finite}} \left(\alpha_1 z^{\beta_1} (\overline{z})^{\beta_2} (\log z)^t + \alpha_2 z^{\gamma_1} (\overline{z})^{\gamma_2} [\log(-\overline{z})]^u \right)$$

$(\alpha_j, \beta_j, \gamma_j \in \mathbb{C},\ t, u \in \mathbb{Z})$; this is not as general as the axial log-polynomial sum as we have defined it, which allows

$$\sum_{\text{finite}} \alpha_{j,t,u,v} z^{\beta_j} \left(\overline{z}\right)^{\beta_j'} (\log z)^t \left[\log\left(-\overline{z}\right)\right]^u (\log iy)^v ,$$

for example. Whether the latter sum ever actually occurs as a period function is as yet unknown. What is certain, though, is that if the period functions considered by Knopp do occur, then so do more general period functions which are obtained from these using the operator $f \to y^w f,\ w \in \mathbb{C}$ [**26**].

7. In keeping with the usual shorthand, we refer to theorems of the type $\mathcal{A} \Rightarrow \mathcal{B}$ as "direct" and $\mathcal{B} \Leftarrow \mathcal{A}$ as "converse" (although Hecke himself did not use this terminology).

5. Applications

Theorem 3.2 has two immediate applications. First we will use it to derive a new proof for an estimate on the growth of the Mellin transform of an automorphic integral (namely, that $\Phi_f(s)$ vanishes faster than any rational function of $\mathrm{Im} s$, as $s \to \pm\infty$ within any vertical strip). Then we will use Theorem 3.2 to disprove a conjecture from [**14**] involving the weight-changing operator $\delta_k = \frac{d}{dz} + \frac{k}{2iy},\ k \in \mathbb{Z}$, which preserves automorphicity on the linear fractional transformation group $\mathcal{G}_\lambda$. Each of these applications further motivates our nonanalytic perspective by providing insight into the analytic milieu.

Theorem 5.1. *(Growth estimate) Let f be an automorphic integral on $\mathcal{H}$ with log-polynomial period function, and let $\Phi_f(s)$ be the Mellin transform of f. Then $\Phi_f(s) = o\left(|\mathrm{Im}\, s|^\rho\right)$ as $\mathrm{Im}\, s \to \pm\infty$, $|\mathrm{Re}\, s| < A$, for any $\rho \in \mathbb{R}$.*

Proof. f is an (analytic) automorphic integral of weight $\alpha \in \mathbb{C}$. Then by [**26**], $\delta^N f$ is a nonanalytic automorphic integral of coweights $\alpha + 2N, 0$ for all $N \in \mathbb{Z}^+ \cup \{0\}$. Theorem 3.1$\Rightarrow$ Φ_f is bounded in the "lacunary vertical strips" $L(\sigma_1, \sigma_2, t_0)$ such that $t_0 > \max|\alpha_j|$ (α_j defined as in Theorem 3.1).

It was observed in [**14**] that for f analytic,

$$\Phi_{\delta f}(s) = i\left(s - 1 - \frac{\alpha}{2}\right)\Phi_f(s-1), \ \forall s \in \mathbb{C}. \tag{5.1}$$

To prove this identity, integrate by parts directly, or else use the functional equation for the gamma function. Although [**14**] deals with integral weights, identity multiplier system and the modular group, we remark that the identity also holds for complex weights, arbitrary multiplier systems (still subject to $v(S_\lambda) = 1$) and all Hecke groups. (Also, the integrals there had rational period functions, but as noted in [**26**], this distinction too is immaterial; the identity still holds if the period functions are log-polynomials.) In fact, when f is analytic, we can easily show that for each $N \in \mathbb{Z}^+ \cup \{0\}$ there exists a polynomial p_N of degree exactly N such that

$$\Phi_{\delta^N f}(s) = p_N(s)\Phi_f(s-N), \ \forall s \in \mathbb{C}. \tag{5.2}$$

(Caution: This is not simply a consequence of inductive application of (5.1), since that identity applied only to analytic f, not to such nonanalytic integrals as $\delta^N f$. Nevertheless, (5.2) is easily proved by induction.)

Now, by Theorem 3.2 with $g = \delta^N f$ we know that $\Phi_{\delta^N f}$ is bounded in lacunary vertical strips with sufficiently large t_0. In fact we may simply take $t_0 > \max|\mathrm{Im}\alpha_j|$ again, where the $\alpha_j's$ are those associated with f and Φ_f, because the poles of Φ_f, $\Phi_{\delta f}$, $\Phi_{\delta^2 f}$, ... *do not increase in imaginary part.* (By linearity of δ, the period function q_N of $\delta^N f$ is $\delta^N q_0$, where q_0 is the period function of f.) If

$$q_0(z) = \sum_{j=1}^{J}\sum_{t=0}^{M_j}\beta_{j,t} z^{\alpha_j}(\log z)^t,$$

then

$$q_N(iy) = \sum_{j=1}^{J'} \sum_{t=0}^{M_j'} \gamma_{j,t} (iy)^{\beta_j} (\log iy)^t ,$$

where $\{\beta_j\} \subseteq \{\alpha_j + \ell \mid 1 \leq j \leq J,\ \ell \in \mathbb{Z}^+\}$. This becomes clear if one applies the product rule from calculus to the individual terms of $q_0(z)$.) By (5.2), then, $p_N(s)\,\Phi_f(s-N)$ is bounded in lacunary vertical strips with large t_0. But this lower bound for t_0 is independent of N, since

$$\max_{\varepsilon_{j,N} = \text{ poles of } \Phi_{\delta^N f}} |\mathrm{Im}\, \varepsilon_{j,N}|$$

is nonincreasing in N. Thus $|\Phi_f(s)| \leq A_N |s|^{-N}$ in those lacunary vertical strips. Since N was arbitrary, $|\Phi_f(s)| = o\left(|s|^{-\rho}\right)\ \forall \rho > 0$, and obviously then this holds $\forall \rho \in \mathbb{R}$. ■

Remark 5.1. *There is actually a stronger estimate on* Φ_f *(*exponential *decay) which is used to prove the converse (*$\mathcal{B} \Rightarrow \mathcal{A}$*) Riemann-Hecke-Bochner Correspondence (Theorem 3.1 Converse); however, that proof relies on Stirling's Formula and the Phragmén-Lindelöf Principle, both of which we have avoided in the proof of Theorem 5.1. Our proof uses only the Direct Theorem (*$\mathcal{A} \Rightarrow \mathcal{B}$*) of Theorem 3.2. Thus, although the estimate applies to analytic automorphic integrals, it can be proved using nonanalytic integrals.*

The second application of Theorem 3.2 deals with the conjecture made in 1983 by Knopp [**14**]. We begin by recounting an earlier result from [**13**]:

Theorem 5.2. *Let* f *be a modular integral with rational period function* q*, weight* $k \in \mathbb{Z}$*, and identity multiplier system. If* q *has poles in* $\mathbb{Q} \cup \{\infty\}$*, then they are in* $\{0, \infty\}$.

(Thus q is a Laurent polynomial, and therefore a log-polynomial sum, which shows the relevance of this theorem to our present situation.) Such period functions were completely characterized in the same paper.

In [**14**], we have the following two theorems:

Theorem 5.3. *Suppose* $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$, $a_n = O(n^{\gamma})$, $\gamma > 0$, *as* $n \to \infty$, *is a modular integral with rational period function* q, *weight* $k \in 2\mathbb{Z}$ *and identity multiplier system, such that the finite poles of* q *are rational (therefore* $= 0$, *by Theorem 5.2). Consider the Mellin transform of* $(\delta_k f)(z) - \frac{a_0}{z}$, *defined by*

$$\Psi(s) = \int_0^{\infty} \left\{ (\delta_k f)(iy) - \frac{a_0 k}{2iy} \right\} y^{s-1} dy.$$

$\Psi(s)$ *has the form*

$$\Psi(s) = (2\pi)^{-s} \left\{ \Gamma(s) - \frac{k}{2} \Gamma(s-1) \right\} \sum_{n=1}^{\infty} b_n n^{-s}$$

and can be continued to a function meromorphic in the entire $s-$*plane, analytic except for finitely many simple poles at rational integer values of* s. *Furthermore,* $\Psi(s)$ *satisfies the functional equation*

$$\Psi(k+2-s) = e^{-\frac{i\pi}{2}(k+2)} \Psi(s). \tag{5.3}$$

Note: Ψ is bounded in all lacunary vertical strips not intersecting $\mathbb{R}$.

Theorem 5.4. *Conversely, suppose*

$$\Phi(s) = (2\pi)^{-s} \Gamma(s) \sum_{n=1}^{\infty} a_n n^{-s},$$

with the Dirichlet series converging in some half-plane. Suppose $\Phi(s)$ *can be continued to a function meromorphic in the* $s-$*plane, analytic except possibly for finitely many simple poles at rational integer values of* s. *Assume also that* $\Phi(s)$ *is bounded*

in every lacunary vertical strip not intersecting the real axis. Let

$$\Psi(s) = i\left(s - 1 - \frac{k}{2}\right)\Phi(s-1),$$

with $k \in 2\mathbb{Z}$ *and suppose* Ψ *satisfies (5.3). Then for any* $a_0 \in C$, Ψ *is the Mellin transform of* $\delta_k f - \frac{a_0 k}{2z}$, *where* $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ *is a modular integral of weight* k, *multiplier system* $\equiv 1$, *with rational period function having poles only at* 0 *and* ∞.

Remark 5.2. *In light of Theorem 5.2, we can view these last two theorems as essentially a special case of Theorem 3.2 with* f *replaced by* $\delta_k f$, $\Psi = \Phi_{\delta_k f}$ *and* $\lambda = 1$. *This fact is nonobvious, since in Theorem 5.4 the hypotheses refer partly to* Ψ *and partly to* Φ, *instead of wholly to* Ψ *as in Theorem 3.2. However, the fact that* $\Phi(s)$ *is meromorphic in* $\mathbb{C}$ *with at worst simple poles in* $\mathbb{Z}$ *implies the same property for* $\Psi(s)$*; also, the boundedness condition on* Φ *implies a seemingly weaker boundedness condition for* Ψ *which is nevertheless equivalent in the presence of the other hypotheses. For,* $|\Psi(s)| \leq A \cdot |s|$ *in lacunary vertical strips, and while this is not quite the same as the statement of Theorem 3.2, the proof works equally well since the step involving application of Stirling's formula results in an estimate on* Ψ *with exponential decay. Thus we can actually allow* polynomial *growth (and not strictly boundedness) on* Ψ *without losing the Converse Theorem.*

Remark 5.3. *This is relevant to the more general setting. Both analytic and nonanalytic Riemann-Hecke-Bochner Correspondence Theorems hold if we replace the boundedness condition on the Mellin transform by polynomial growth, although of course it must follow, then, that the stronger boundedness condition holds in the presence of the functional equation and meromorphicity condition.*

Thus, in Theorem 5.4 , we have

$$\begin{aligned}\Psi(s) &= i\left(s-1-\tfrac{k}{2}\right)\Phi(s-1)\\ &= i\left(\tfrac{\Gamma(s)}{\Gamma(s-1)}-\tfrac{k}{2}\right)\Phi(s-1)\\ &= i\left(\tfrac{\Gamma(s)}{\Gamma(s-1)}-\tfrac{k}{2}\right)(2\pi)^{-s+1}\Gamma(s-1)\sum_{n=1}^{\infty}a_n n^{-s+1}\\ &= (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}(2\pi i n a_n)n^{-s}+(2\pi)^{-(s-1)}\Gamma(s-1)\sum_{n=1}^{\infty}\tfrac{-ik}{2}a_n n^{-(s-1)},\end{aligned}$$

so applying Theorem 3.2 ($\mathcal{B}\Rightarrow\mathcal{A}$) to Ψ, we do indeed obtain the conclusions of Theorem 5.4.

As for Theorem 5.3, here δf has coweights $\alpha = k+2\in 2\mathbb{Z}$, $\beta = 0$ and period function $q(z)$ with $q(iy)=\sum_{\ell=-L}^{L} d_\ell (iy)^\ell$, so in the notation of Theorem 3.2 we have $M_j\equiv 0$, $\{\alpha_j\}\subseteq\mathbb{Z}$, $M=1$, $w_1=0$, and $\alpha,\beta\in\mathbb{Z}$. Thus by Theorem 3.2 and the corollary, Ψ has the requisite meromorphic continuation, its poles are simple (since $M_j\equiv 0$) and they are in $\mathbb{Z}$ (since α,β,w_m, and α_j are all rational integers). That $\Psi(s)=(2\pi)^{-s}\left\{\Gamma(s)-\frac{k}{2}\Gamma(s-1)\right\}\sum_{n=1}^{\infty}b_n n^{-s}$ is a simple calculation.

In the same reference [**14**] we find the following

Conjecture. *Theorem 5.4 still holds if we assume only that*

$$\Psi(s)=(s-1+\mu)\Phi(s-1),$$

and not necessarily that $\Psi(s)=\left(s-1-\frac{k}{2}\right)\Phi(s-1)$; *one can show that* $\mu=-\frac{k}{2}$ *and then simply apply the earlier result (Theorem 5.4).*

We will disprove this conjecture, by way of the following counterexample.

Example 5.1. *Let* $k\in 2\mathbb{Z}^+$, $\mu=0$ $(\neq -\frac{k}{2})$ *and put* $f(z)=a_0+\sum_{n=1}^{\infty}a_n e^{2\pi i n z}$, *with* a_0 *a complex number and*

$$a_n = 2\frac{(2\pi i)^{k+1}}{n(k+1)!}\sigma_{k+1}(n)$$

for $n\geq 1$, *where* $\sigma_{k+1}(n)=\sum_{d|n,\ d>0} d^{k+1}$. *Then,*

$$(\delta_{-2\mu} f)(z) = (\delta_0 f)(z) = f'(z) = 2\tfrac{(2\pi i)^{k+2}}{(k+1)!} \sum_{n=1}^{\infty} \sigma_{k+1}(n)\, e^{2\pi i n z}$$
$$= G_{k+2} - 2\zeta(k+2),$$

which is a modular integral of weight $k+2$, $\upsilon \equiv 1$ *and trivial period function. ('Trivial' in this sense means the modular integral differs from a modular form by a constant term only.)* G_t *denotes the Eisenstein series of coweights* $t, 0$*:*

$$G_t(z) = \sum_{c,d\in\mathbb{Z},\ (c,d)\neq(0,0)} (cz+d)^{-t}.$$

Define the "slash" operator $|_{\alpha,\beta}$ *by*

$$h\,|_{\alpha,\beta}\, M = (cz+d)^{-\alpha} (c\overline{z}+d)^{-\beta} h(-1/z),$$

for h *defined on* $\mathcal{H}$ *and* M *having lower row* c, d*. Since* $G_{k+2}\,|^1_{k+2,0}\, T = G_{k+2}$*, we have*

$$(\delta_{-2\mu} f)\,|^1_{k+2,0}\, T = \delta_{-2\mu} f + 2\zeta(k+2)\left(1 - z^{-k-2}\right),$$

so by Theorem 3.2 the Mellin transforms Φ, Ψ *of* $f, \delta_{-2\mu} f$ *satisfy the hypotheses of the conjecture; but* f *is not a modular integral of weight* k*, identity multiplier and Laurent polynomial period function. For, if that were true, then we would have* $f\,|^1_{k,0}\, T = f + \sum_{\ell=-L}^{L} c_\ell z^\ell$*, and so* $z^{-k} f\left(\frac{-1}{z}\right) = f(z) + \sum_{\ell=-L}^{L} c_\ell z^\ell$*. Differentiate to get* $z^{-k-2} f'\left(\frac{-1}{z}\right) - k z^{-k-1} f\left(\frac{-1}{z}\right) = f'(z) + \sum_{\ell=-L}^{L} \ell c_\ell z^{\ell-1}$*. Since* $f' = G_{k+2} - 2\zeta(k+2)$*, we have* $z^{-k-2} G_{k+2}\left(\frac{-1}{z}\right) - 2z^{-k-2}\zeta(k+2) - k z^{-k-1} f\left(\frac{-1}{z}\right) = G_{k+2}(z) - 2\zeta(k+2) + \sum_{\ell=-L}^{L} \ell c_\ell z^{\ell-1}$*. Therefore,* $-k z^{-k-1} f\left(\frac{-1}{z}\right) = \sum_{\ell=-L}^{L} \ell c_\ell z^{\ell-1} + 2\left(z^{-k-2} - 1\right)\zeta(k+2)$.

Replacing z *by* $\frac{-1}{z}$*, we get that* f*, and therefore* $f' = G_{k+2}$*, is a Laurent polynomial. But this is impossible since* G_{k+2} *is a nonconstant periodic function. Thus we obtain the desired contradiction, and so the conjecture fails in every positive even weight.*

Actually, we could have used Theorem 3.1 (analytic Hecke correspondence), not Theorem 3.2 (nonanalytic Hecke correspondence), here, since we are taking the Mellin transform of the analytic function $\delta_{-2\mu} f = G_{k+2}$. However it should be noted that we discovered this example using Theorem 3.2 ($\mathcal{B} \Rightarrow \mathcal{A}$) to narrow down the possible options for a counterexample by increasing the number of conditions such an example would have to satisfy. In effect, the conditions on Φ and Ψ (namely, their functional equations) necessitated $(\delta_{-2\mu} f) \mid^1_{k+2,0} T = \delta_{-2\mu} f$ without having $\mu = \frac{-k}{2}$; this is exactly what led to the counterexample given here, i.e. using a function which is an antiderivative of a modular integral.

Strictly speaking, the correspondence due to Bochner deals with a more general class of exponential series than that described in Theorem 3.1; namely, it admits two functions f and g satisfying

$$z^{-k} g(-1/z) - C f(z) = \text{log-polynomial sum}.$$

Here we state an analogous theorem for nonanalytic functions. Its proof is omitted since it is virtually identical to that of Theorem 3.2. It is patterned after the analytic version given in [**19, 20**].

Theorem 5.6 (Nonanalytic Hecke Correspondence for Two Functions). *Let* $\lambda_1,\ \lambda_2,\ \gamma > 0$*;* $C,\ \alpha,\ \beta \in \mathbb{C}$*;* $\{w_m\}_{m=1}^{M_1},\ \{v_m\}_{m=1}^{M_2},\ \{a_{n_1,n_2,m} \mid 1 \le m \le M_1,\ 0 \le n_1, n_2 < \infty\},\ \{b_{n_1,n_2,m} \mid 1 \le m \le M_2, 0 \le n_1, n_2 < \infty\} \subseteq \mathbb{C}$ *with* w_m *distinct,* v_m *distinct and*

$$\sum_{n_1+n_2=n} |a_{n_1,n_2,m}|,\ \sum_{n_1+n_2=n} |b_{n_1,n_2,m}| = O(n^{\gamma}) \ \text{ as } n \to \infty$$

for each fixed m *(*γ *is independent of* m*). Define*

$$\begin{aligned} c_{n,m} &= \sum_{n_1+n_2=n} a_{n_1,n_2,m}, \\ d_{n,m} &= \sum_{n_1+n_2=n} b_{n_1,n_2,m}, \end{aligned}$$

for $n \in \mathbb{Z}^+ \cup \{0\}$. *Put*

$$f(z) = \sum_{m=1}^{M_1} y^{w_m} \sum_{n_1,n_2=0}^{\infty} a_{n_1,n_2,m} \exp\left\{\frac{2\pi i}{\lambda_1}(n_1 z - n_2 \overline{z})\right\},$$

$$g(z) = \sum_{m=1}^{M_2} y^{v_m} \sum_{n_1,n_2=0}^{\infty} b_{n_1,n_2,m} \exp\left\{\frac{2\pi i}{\lambda_2}(n_1 z - n_2 \overline{z})\right\},$$

and let $\Phi(s)$ *and* $\Psi(s)$ *be defined as their respective Mellin transforms*

$$\begin{aligned} \Phi(s) &= \int_0^{\infty} \left\{ f(iy) - \sum_{m=1}^{M_1} a_{0,0,m} y^{w_m} \right\} y^s \frac{dy}{y}, \\ \Psi(s) &= \int_0^{\infty} \left\{ g(iy) - \sum_{m=1}^{M_2} b_{0,0,m} y^{v_m} \right\} y^s \frac{dy}{y}, \end{aligned}$$

for $\mathrm{Re}\, s$ *large. (Thus,* $\Phi(s) = \sum_{m=1}^{M_1} \left(\frac{2\pi}{\lambda_1}\right)^{-s-w_m} \Gamma(s+w_m) \sum_{n=1}^{\infty} c_{n,m} n^{-s-w_m}$ *and* $\Psi(s) = \sum_{m=1}^{M_2} \left(\frac{2\pi}{\lambda_2}\right)^{-s-v_m} \Gamma(s+v_m) \sum_{n=1}^{\infty} d_{n,m} n^{-s-v_m}$.*)*

Then the following are equivalent:

(A) $z^{-\alpha} (\overline{z})^{-\beta} g\left(\frac{-1}{z}\right) = C f(z) +$ *axial log-polynomial sum.*

(B) Φ, Ψ *have meromorphic continuations to* $\mathbb{C}$, *with at most a finite number of poles; are bounded in lacunary strips (with the same restriction as before on* t_0, *as in Theorem 3.2); and satisfy* $\Psi(\alpha+\beta-s) = i^{\alpha-\beta} C \Phi(s)$.

Although the Riemann-Hecke-Bochner Correspondence is often given this sort of two-function setting, it is usually applied in the case $f = g$ (Theorem 3.1). Here we give an example where $f \neq g$:

It is easy to see that

$$\sum_{c,d \in \mathbb{Z},\ (c,d) \neq (0,0)} (cz+d)^{-k} \equiv 0$$

for k odd, $k \geq 3$. However, if we consider instead

$$\begin{aligned} \widehat{G}_k(z) &= \sum_{c=1}^{\infty} \sum_{d=-\infty}^{\infty} (cz+d)^{-k}, \\ \widehat{\widehat{G}}_k(z) &= \sum_{c=-\infty}^{\infty} \sum_{d=1}^{\infty} (cz+d)^{-k}, \end{aligned}$$

then we have

$$z^{-k} \widehat{G}_k\left(\frac{-1}{z}\right) = -\widehat{\widehat{G}}_k(z).$$

This shows that $\widehat{G}_k$, $\widehat{\widehat{G}}_k$ satisfy the hypotheses of the last theorem with $\alpha = k$, $\beta = 0$, $\lambda_1 = \lambda_2 = 1$ and that *(A)* holds with $C = -1$ and axial log-polynomial sum $= 0$.

For an example with α, β nonzero, one may simply generalize to an example analogous to the Maass nonanalytic Eisenstein series which appears in [**22**], amending the double-sum to the appropriate subsum as before.

6. Proof of Theorem 3.2

First we will require a lemma.

Lemma 6.1. *With f, q as in Theorem 3.2, assume also that*

$$f(iy) \not\equiv \sum_{m=1}^{M} c_{o,m} y^{w_m}$$

for $y > 0$. Then $C = \pm i^{\beta-\alpha}$, and

$$Ci^{2(\alpha-\beta)} q(z) + (-z)^{-\alpha} (-\overline{z})^{-\beta} q(-1/z) = Ci^{2(\alpha-\beta)} \widehat{q}(z) + (-z)^{-\alpha} z^{-\beta} \widehat{q}(-1/z) = 0$$

for all $z \in \mathcal{H}$, where $\widehat{q}$ is the (unique) analytic function on $\mathcal{H}$ with the property that $q(iy) = \widehat{q}(iy)$ $\forall y > 0$. (I.e., q is a bona fide log-polynomial sum.)

Proof. In the classical case, where q is analytic, this lemma is easier to prove; one merely notes that a periodic log-polynomial sum is constant, and the desired conclusion follows directly. Our nonanalytic q will require greater care.

Replacing z by $\frac{-1}{z}$ in the transformation law for f, we obtain

$$Cf\left(-1/z\right)+q\left(-1/z\right)=\left(-1/z\right)^{-\alpha}\overline{\left(-1/z\right)}^{(-\beta)}f\left(z\right)=\left(-z\right)^{\alpha}\left(-\overline{z}\right)^{\beta}f\left(z\right).$$

Thus,

$$\begin{aligned}f\left(z\right)&=\left(-z\right)^{-\alpha}\left(-\overline{z}\right)^{-\beta}\left[Cf\left(-1/z\right)+q\left(-1/z\right)\right]\\&=\left(-z\right)^{-\alpha}\left(-\overline{z}\right)^{-\beta}\left[Cz^{\alpha}\overline{z}^{(\beta)}\left(Cf\left(z\right)+q\left(z\right)\right)+q\left(-1/z\right)\right]\\&=Ci^{2(\alpha-\beta)}\left[Cf\left(z\right)+q\left(z\right)\right]+\left(-z\right)^{-\alpha}\left(-\overline{z}\right)^{-\beta}q\left(-1/z\right),\end{aligned}$$

for all $z\in\mathcal{H}$, $\alpha,\beta\in\mathbb{C}$. (For $\mathrm{Im}z>0$, $\left(-z\right)^{-\alpha}\left(-\overline{z}\right)^{-\beta}z^{\alpha}\overline{z}^{(\beta)}=i^{2(\alpha-\beta)}$, by the Open Mapping Theorem and its analog for conjugate-analytic functions.)

It follows that

$$\left[1-C^{2}i^{2(\alpha-\beta)}\right]f\left(z\right)=Ci^{2(\alpha-\beta)}q\left(z\right)+\left(-z\right)^{-\alpha}\left(-\overline{z}\right)^{-\beta}q\left(-1/z\right),\tag{6.1}$$

for $\mathrm{Im}z>0$. Put $z=iy$. Then,

$$\left[1-C^{2}i^{2(\alpha-\beta)}\right]f\left(iy\right)=Ci^{2(\alpha-\beta)}q\left(iy\right)+\left(-iy\right)^{-\alpha}\left(iy\right)^{-\beta}q\left(-1/iy\right),$$

$\forall y>0$. By extending both sides of (6.1) analytically to $\mathcal{H}$, we get

$$\left[1-C^{2}i^{2(\alpha-\beta)}\right]h\left(z\right)=Ci^{2(\alpha-\beta)}\widehat{q}\left(z\right)+\left(-z\right)^{-\alpha}z^{-\beta}\widehat{q}\left(-1/z\right)\text{ for all }z\in\mathcal{H},\tag{6.2}$$

where

$$h\left(z\right)=\sum_{m=1}^{M}\left(-iz\right)^{w_{m}}\sum_{n=0}^{\infty}c_{n,m}e^{2\pi inz/\lambda}.$$

Therefore,

$$\left[1-C^{2}i^{2(\alpha-\beta)}\right]\sum_{m=1}^{M}\left(-iz\right)^{w_{m}}\sum_{n=1}^{\infty}c_{n,m}e^{2\pi inz/\lambda}\tag{6.3}$$

$$= Ci^{2(\alpha-\beta)}\widehat{q}(z) + (-z)^{-\alpha} z^{-\beta}\widehat{q}(-1/z) - \left[1 - C^2 i^{2(\alpha-\beta)}\right] \sum_{m=1}^{M} c_{o,m} (-iz)^{w_m},$$

for all $z \in \mathcal{H}$.

The right-hand side of (6.3) is a log-polynomial sum, while the left-hand side decays exponentially as $z \to \infty$ within any set of the form

$$W_\theta = \left\{ z \in \mathbb{C} : z \neq 0,\ \left| \frac{\pi}{2} - \arg z \right| < \theta \right\},\ 0 \leq \theta \leq \frac{\pi}{2}.$$

It is proved in [**26**] that any log-polynomial sum which satisfies such a limiting condition must be *identically* zero. But

$$\sum_{m=1}^{M} (-iz)^{w_m} \sum_{n=1}^{\infty} c_{n,m} e^{2\pi i n z/\lambda} \not\equiv 0 \text{ in } \mathcal{H},$$

since, by hypothesis,

$$\sum_{m=1}^{M} y^{w_m} \sum_{n=1}^{\infty} c_{n,m} e^{-2\pi n y/\lambda} = f(iy) - \sum_{m=1}^{M} c_{o,m} y^{w_m} \not\equiv 0 \text{ in } \mathbb{R}^+.$$

Thus, (6.3) implies that $1 - C^2 i^{2(\alpha-\beta)} = 0$, which gives the first conclusion of the lemma. This, in turn, implies the rest of the lemma, by (6.1) and (6.2). ■

6.1. **Proof of the Direct Theorem.** With Lemma 6.1 in hand, we may begin the proof of Theorem 3.2, $\mathcal{A} \Rightarrow \mathcal{B}$.

Assume $\mathcal{A}$. If $f(iy) \equiv \sum_{m=1}^{M} c_{o,m} y^{w_m}$ for $y > 0$, then $\Phi_f(s) \equiv 0$ and $\mathcal{B}$ holds trivially. Thus, we may assume $f(iy) \not\equiv \sum_{m=1}^{M} c_{o,m} y^{w_m}$ for $y > 0$.

That Φ_f is the Mellin transform of $f - \sum_{m=1}^{M} c_{o,m} i^{-w_m} z^{w_m} = f - \sum_{m=1}^{M} c_{o,m} y^{w_m}$ (as stated in remarks following Theorem 3.2) is a consequence of Fubini's Theorem.

For $\mathrm{Re}\, s$ large, then,

$$\Phi_f(s) = \int_0^\infty \left[f(iy) - \sum_{m=1}^{M} c_{o,m} y^{w_m} \right] y^{s-1} dy = \int_1^\infty + \int_0^1.$$

Taking $y \mapsto \dfrac{1}{y}$ in the latter integral and then applying the transformation law for f, we get

$$\begin{aligned}
\Phi_f(s) &= \int_1^\infty \left[f(iy) - \sum_{m=1}^M c_{o,m} y^{w_m} \right] y^{s-1} dy + \\
&\qquad + \int_1^\infty \left\{ (iy)^\alpha (-iy)^\beta [Cf(iy) + q(iy)] - \sum_{m=1}^M c_{o,m} y^{-w_m} \right\} y^{-s-1} dy \\
&= \int_1^\infty \left[f(iy) - \sum_{m=1}^M c_{o,m} y^{w_m} \right] y^{s-1} dy + i^{\alpha-\beta} \int_1^\infty \left[f(iy) - \sum_{m=1}^M c_{o,m} y^{w_m} \right] y^{\alpha+\beta-s-1} dy \\
&\quad + i^{\alpha-\beta} \int_1^\infty q(iy)\, y^{\alpha+\beta-s-1} dy + \sum_{m=1}^M c_{o,m} \left(i^{\alpha-\beta} C \int_1^\infty y^{w_m+\alpha+\beta-s-1} dy - \int_1^\infty y^{-w_m-s-1} dy \right) \\
&= \int_1^\infty \left[f(iy) - \sum_{m=1}^M c_{o,m} y^{w_m} \right] y^{s-1} dy + i^{\alpha-\beta} \int_1^\infty \left[f(iy) - \sum_{m=1}^M c_{o,m} y^{w_m} \right] y^{\alpha+\beta-s-1} dy \\
&\quad + i^{\alpha-\beta} \int_1^\infty q(iy)\, y^{\alpha+\beta-s-1} dy + \sum_{m=1}^M c_{o,m} \left[\tfrac{i^{\alpha-\beta} C}{s-(\alpha+\beta+w_m)} - \tfrac{1}{s+w_m} \right].
\end{aligned}$$

(Each of these integrals converges for $\operatorname{Re} s$ sufficiently large.) Therefore $\Phi_f = E + L + r$, where

$$E(s) = \int_1^\infty \left[f(iy) - \sum_{m=1}^M c_{o,m} y^{w_m} \right] y^{s-1} dy + i^{\alpha-\beta} \int_1^\infty \left[f(iy) - \sum_{m=1}^M c_{o,m} y^{w_m} \right] y^{\alpha+\beta-s-1} dy,$$

$$L(s) = i^{\alpha-\beta} \int_1^\infty q(iy)\, y^{\alpha+\beta-s-1} dy$$

and

$$r(s) = \sum_{m=1}^M c_{o,m} \left[\frac{i^{\alpha-\beta} C}{s-(\alpha+\beta+w_m)} - \frac{1}{s+w_m} \right].$$

Since $E(s)$ is absolutely uniformly convergent on compact subsets of $\mathbb{C}$, it is entire, by the integral analog of the Weierstrass M-test [**29**, p. 100]. Now, for $\operatorname{Re} s$ large,

$$L(s) = i^{\alpha-\beta} \int_1^\infty \sum_{j=1}^J (iy)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{j,t} (\log iy)^t\, y^{\alpha+\beta-s-1} dy.$$

The integral and the finite double-sum may be switched, since each term is absolutely integrable. Writing $\log iy = \frac{i\pi}{2} + \log y$ and applying the binomial theorem, we obtain

$$L(s) = \sum_{j=1}^J i^{\alpha_j+\alpha-\beta} \sum_{t=0}^{M_j} \beta_{j,t} \sum_{\ell=0}^t \binom{t}{\ell} \left(\frac{i\pi}{2} \right)^{t-\ell} \int_1^\infty y^{\alpha_j+\alpha+\beta-s-1} (\log y)^\ell\, dy.$$

Observe that

$$\int_1^\infty y^\eta (\log y)^\ell = (-1)^\ell \ell! (\eta+1)^{-\ell-1} \tag{6.4}$$

for $\ell \in Z^+ \cup \{0\}$ and $\mathrm{Re}\eta < -1$, so that

$$L(s) = \sum_{j=1}^J i^{\alpha_j+\alpha-\beta} \sum_{t=0}^{M_j} \beta_{j,t} \sum_{\ell=0}^t \binom{t}{\ell} \left(\frac{i\pi}{2}\right)^{t-\ell} \frac{(-1)^{\ell+1}\ell!}{(\alpha_j+\alpha+\beta-s)^{\ell+1}} \tag{6.5}$$

for $\mathrm{Re}s > \mathrm{Re}(\alpha-\beta) + \max \mathrm{Re}\alpha_j$. This shows that L is meromorphic in $\mathbb{C}$ with a pole of order M_j+1 at $\alpha_j+\alpha+\beta$. (We assume, as always, that $\beta_{j,M_j} \neq 0$ for all j.) This proves part (i) of $\mathcal{B}$ in Theorem 3.2.

Next it will be shown that Φ_f satisfies the functional equation $\Phi_f(\alpha+\beta-s) = i^{\alpha-\beta}C\Phi_f(s)$, $s \in \mathbb{C}$. Clearly E and r satisfy this same functional equation, by their very definitions; therefore it suffices to consider L $(= \Phi_f - E - r)$. By Lemma 6.1 $(z = iy)$, we have $\left(i^{\alpha-\beta}C\right)^2 = 1$ and $Ci^{2(\alpha-\beta)}q(iy) + (-iy)^{-\alpha}(iy)^{-\beta}q(-1/iy) = 0$, so that $q(-1/iy) = -Ci^{\beta-\alpha}y^{\alpha+\beta}q(iy)$. By the definition of L, then,

$$\begin{aligned}
i^{\alpha-\beta}C \cdot L(\alpha+\beta-s) &= i^{2(\alpha-\beta)}C\int_1^\infty q(iy)\, y^{s-1}dy \\
&= i^{2(\alpha-\beta)}C\int_1^0 q(i/y)\, y^{-s+1}\tfrac{dy}{-y^2} \\
&= i^{2(\alpha-\beta)}C\int_0^1 q(i/y)\, y^{-s-1}dy \\
&= i^{2(\alpha-\beta)}C\int_0^1 q(-1/iy)\, y^{-s-1}dy \\
&= -Ci^{\beta-\alpha}\int_0^1 q(-1/iy)\, y^{\alpha+\beta-s-1}dy, \text{ by Lemma 6.1.}
\end{aligned}$$

$$\begin{aligned}
\text{Thus } i^{\alpha-\beta}C \cdot L(\alpha+\beta-s) &= -i^{\alpha-\beta}\int_0^1 \sum_{j=1}^J (iy)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{j,t} (\log iy)^t\, y^{\alpha+\beta-s-1}dy \\
&= -i^{\alpha-\beta}\sum_{j=1}^J i^{\alpha_j} \sum_{t=0}^{M_j} \beta_{j,t}\int_0^1 y^{\alpha_j} (\log iy)^t\, y^{\alpha+\beta-s-1}dy,
\end{aligned}$$

where the interchange of integral and sum is justified as before.

Writing $\log iy = \frac{i\pi}{2} + \log y$, substituting $y \mapsto \frac{1}{y}$ and applying (6.4) again, we get

$$i^{\alpha-\beta}C\cdot L\left(\alpha+\beta-s\right)=i^{\alpha-\beta}\sum_{j=1}^{J}i^{\alpha_j}\sum_{t=1}^{M_j}\beta_{j,t}\sum_{\ell=1}^{t}\tbinom{t}{\ell}\left(\tfrac{i\pi}{2}\right)^{t-\ell}\ell!\left(s-\alpha_j-\alpha-\beta\right)^{-\ell-1},$$

for $\operatorname{Re}s$ small; and since L is meromorphic in the plane, the preceding equation holds in $\mathbb{C}$. Comparing the earlier expression, one sees readily that

$$i^{\alpha-\beta}C\cdot L\left(\alpha+\beta-s\right)=L\left(s\right).$$

This verifies the functional equation for L, and thus for Φ_f, proving (iii).

Finally, we come to the boundedness condition (ii). The rational functions L and r are, of course, bounded in (closed) lacunary vertical strips which do not contain poles of L or r. We have demonstrated that said poles are in $\{-\alpha_j\}\cup\{-w_m\}\cup\{\alpha+\beta+w_m\}$. Thus, to prove (ii), it suffices to confirm the boundedness of $E\left(s\right)$. This is simple: if $s\in L\left(\sigma_1,\sigma_2,t_0\right)$, then

$$\begin{aligned}
&\left|\int_1^\infty\left[f\left(iy\right)-\sum_{m=1}^{M}c_{o,m}y^{w_m}\right]y^{s-1}dy\right|\\
&\quad\leq\sum_{m=1}^{M}\int_1^\infty\sum_{n_1,n_2=0}^{\infty}\left[\left|a_{n_1,n_2,m}\right|e^{-2\pi(n_1+n_2)y/\lambda}-\left|a_{0,0,m}\right|\right]y^{\operatorname{Re}(w_m+s)-1}dy\\
&\quad=\sum_{m=1}^{M}\sum_{n=1}^{\infty}\left(\sum_{n_1+n_2=n}\left|a_{n_1,n_2,m}\right|\right)\int_1^\infty e^{-2\pi ny/\lambda}y^{\operatorname{Re}(w_m+s)-1}dy\\
&\quad\leq A_1\sum_{m=1}^{M}\sum_{n=1}^{\infty}n^{\gamma}\int_1^\infty e^{-2\pi ny/\lambda}y^{\operatorname{Re}w_m+\sigma_2-1}dy<\infty.
\end{aligned}$$

This proves part (ii) of $\mathcal{B}$ and completes the proof of the direct theorem.

6.2. **Proof of the Corollary.** We are now in a position to prove the corollary. From the definition of L and the functional equation for q (Lemma 6.1),

$$L\left(s\right)=-i^{2(\alpha-\beta)}C^{-1}\int_1^\infty q\left(-1/iy\right)y^{-s-1}dy.$$

By calculations similar to those used previously, then,

$$L\left(s\right)=-i^{2(\alpha-\beta)}C^{-1}\sum_{j=1}^{J}i^{\alpha_j}\sum_{t=1}^{M_j}\beta_{j,t}\sum_{\ell=1}^{t}\binom{t}{\ell}\left(\frac{i\pi}{2}\right)^{t-\ell}\ell!\left(-\alpha_j-s\right)^{-\ell-1},\tag{6.6}$$

for $\operatorname{Re}s$ small. Compare (6.5). Since $E=\Phi_f-L-r$ is entire, this proves the first part of the corollary.

Equations (6.5) and (6.6) also imply the set identity $\{-\alpha_j\}_{j=1}^{J} = \{\alpha_j + \alpha + \beta\}_{j=J}^{J}$. Since the α_j are distinct, then, we have the following:

- If J is even, $\exists \pi \in S_J$ such that $\alpha_{\pi(j+1)} = -\alpha_{\pi(j)} - \alpha - \beta$ for $j = 1, 3, ..., J-1$.
- If J is odd, $\exists \pi \in S_J$ such that $\alpha_{\pi(j+1)} = -\alpha_{\pi(j)} - \alpha - \beta$ for $j = 1, 3, ..., J-2$ and $\alpha_{\pi(J)} = -\alpha_{\pi(J)} - \alpha - \beta$ (i.e. $\alpha_{\pi(J)} = -\frac{1}{2}(\alpha - \beta)$).

with S_J denoting the symmetric group on J letters. (Also, of course, $\beta_{j,t}$ and $\beta_{\pi(j),t}$ are related.) Thus the exponent set of any axial log-polynomial period function is

$$\{\alpha_j\}_{j=1}^{J} = \left\{\eta_1, \eta_2, \ldots, \eta_{\lfloor J/2 \rfloor}\right\} \cup \left\{\eta_1 - \alpha - \beta, \eta_2 - \alpha - \beta, \ldots, \eta_{\lfloor J/2 \rfloor} - \alpha - \beta\right\} \cup S,$$

$$\text{where } S = \begin{cases} \emptyset & \text{if } J \text{ is even,} \\ \left\{-\frac{\alpha+\beta}{2}\right\} & \text{if } J \text{ is odd.} \end{cases}$$

This is actually a slightly stronger statement than was needed, and so this completes the proof of the corollary.

6.3. **Proof of the Converse Theorem.** It remains to be shown that $\mathcal{B} \Rightarrow \mathcal{A}$. Using the identity $e^{-y} = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} y^{-s} \Gamma(s)\, ds$, $d > 0$ [**21**] and a standard calculation, we can show that f is the inverse Mellin transform of Φ_f. For $y > 0$ and d large, then, $f(iy) - \sum_{m=1}^{M} c_{o,m} y^{w_m} = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Phi_f(s)\, y^{-s} ds = \frac{i^{\beta-\alpha}}{2\pi i C} \int_{d-i\infty}^{d+i\infty} \Phi_f(\alpha + \beta - s)\, y^{-s} ds$

$= \frac{i^{\beta-\alpha}}{2\pi i C} \int_{\alpha+\beta-d+i\infty}^{\alpha+\beta-d-i\infty} \Phi_f(s)\, y^{s-(\alpha+\beta)} d(-s) = \frac{i^{\beta-\alpha}}{2\pi i C} \cdot y^{-\alpha-\beta} \int_{\alpha+\beta-d-i\infty}^{\alpha+\beta-d+i\infty} \Phi_f(s)\, y^s ds.$

Therefore,

$$f(iy) - \sum_{m=1}^{M} c_{o,m} y^{w_m} = \frac{1}{2\pi i C} \cdot (iy)^{-\alpha} (-iy)^{-\beta} \int_{\alpha+\beta-d-i\infty}^{\alpha+\beta-d+i\infty} \Phi_f(s)\, y^s ds. \tag{6.7}$$

Now,

$$\int_{\alpha+\beta-d-i\infty}^{\alpha+\beta-d+i\infty} \Phi_f(s)\, y^s ds = \lim_{T\to\infty} \left[\int_{d-iT}^{d+iT} + \int_{\gamma_1(T)} + \int_{\gamma_2(T)} \right] - 2\pi i \cdot \sum_{s_\nu} Res\left(y^s \Phi_f; s_\nu\right) \tag{6.8}$$

(see Figure 2), choosing d and T large enough for the parallelogram shown to enclose all of the poles s_ν of Φ_f; recall that these are finite in number. (N.B.: In the classical case, the contour is *rectangular*.)

For d large enough, both $\int\limits_{\gamma_1(T)}$ and $\int\limits_{\gamma_2(T)} \to 0$ as $T \to \infty$, by $\mathcal{B}$ (ii) and (iii), together with the Phragmén-Lindelöf theorem [**28**] and Stirling's formula. (To satisfy the hypotheses of the former, we must consider instead of Φ_f the auxiliary function $e^{-\pi i s/4}\Phi_f$. This again contrasts with the proof of Theorem 3.1.)

By (6.7) and (6.8), then,

$$f(iy) - \sum_{m=1}^{M} c_{o,m} y^{w_m}$$
$$= \tfrac{1}{2\pi i C} (iy)^{-\alpha} (-iy)^{-\beta} \left[\int\limits_{d-i\infty}^{d+i\infty} \Phi_f(s)\, y^s ds - 2\pi i \cdot \sum_{\text{poles } s_\nu \text{ of } \Phi_f} Res\left(y^s \Phi_f; s_\nu\right) \right].$$

But $f(iy) - \sum\limits_{m=1}^{M} c_{o,m} y^{w_m} = \frac{1}{2\pi i} \int\limits_{d-i\infty}^{d+i\infty} \Phi_f(s)\, y^{-s} ds$ for $y > 0$, so

$f\left(\frac{-1}{iy}\right) - \sum\limits_{m=1}^{M} c_{o,m} y^{w_m} = \frac{1}{2\pi i} \int\limits_{d-i\infty}^{d+i\infty} \Phi_f(s)\, y^s ds$. Thus,

$$f(iy) - \sum_{m=1}^{M} c_{o,m} y^{w_m}$$
$$= C^{-1} (iy)^{-\alpha} (-iy)^{-\beta} \left[f\left(\tfrac{-1}{iy}\right) - \sum_{m=1}^{M} c_{o,m} y^{w_m} \right] - (iy)^{-\alpha} (-iy)^{-\beta} \cdot \sum_{s_\nu} Res\left(y^s \Phi_f; s_\nu\right).$$

Say the pole set of Φ_f is $\{s_\nu\}_{\nu=1}^{V}$, and that the principal part of Φ_f at $s = s_\nu$ is $\sum\limits_{u=1}^{A_\nu} \eta_{\nu,u} (s - s_\nu)^{-u}$. The residue of $y^s \Phi_f(s)$ at $s = s_\nu$ is $\sum\limits_{u=1}^{A_\nu} y^{s_\nu} \eta_{\nu,u} \frac{(\log y)^{u-1}}{(u-1)!}$, and so $\sum\limits_{\nu=1}^{V} Res\left(y^s \Phi_f; s_\nu\right) = \sum\limits_{\nu=1}^{V} \sum\limits_{u=1}^{A_\nu} y^{s_\nu} \eta_{\nu,u} \frac{(\log y)^{u-1}}{(u-1)!}$. Thus,

$$f(iy) - \sum_{m=1}^{M} c_{0,m} y^{w_m} = (iy)^{-\alpha} (-iy)^{-\beta} C^{-1} \left[f\left(\tfrac{-1}{iy}\right) - \sum_{m=1}^{M} c_{0,m} y^{-w_m} \right]$$
$$- (iy)^{-\alpha} (-iy)^{-\beta} C^{-1} \sum_{\nu=1}^{V} \sum_{u=1}^{A_\nu} y^{s_\nu} \eta_{\nu,u} \tfrac{(\log y)^{u-1}}{(u-1)!}.$$

Therefore,

$$(iy)^{-\alpha} (-iy)^{-\beta} f\left(\tfrac{-1}{iy}\right) - C f(iy) = -\sum_{m=1}^{M} c_{0,m} y^{w_m} + (iy)^{-\alpha} (-iy)^{-\beta} \sum_{m=1}^{M} c_{0,m} y^{-w_m}$$
$$+ i^{\beta-\alpha} (iy)^{-\alpha} (-iy)^{-\beta} \sum_{\nu=1}^{V} \sum_{u=1}^{A_\nu} y^{s_\nu} \eta_{\nu,u} \tfrac{(\log y)^{u-1}}{(u-1)!},$$

and since $y^{s_\nu} = (-i)^{s_\nu}(iy)^{s_\nu}$ and $\log y = \log iy - \frac{i\pi}{2}$ for $y > 0$, we have that $(iy)^{-\alpha}(-iy)^{-\beta} f\left(\frac{-1}{iy}\right) - Cf(iy)$ is equal to a log-polynomial sum on $i\mathbb{R}^+$, so $z^{-\alpha}(\overline{z})^{-\beta} f\left(\frac{-1}{z}\right) - Cf(z)$ is equal to an axial log-polynomial sum on $\mathcal{H}$. Thus $\mathcal{A}$ holds. (In the holomorphic case, one effects an analytic continuation from $i\mathbb{R}^+$ to $\mathcal{H}$; for *axial* sums this is both impossible and, fortunately, unnecessary.)

This completes the proof of Theorem 3.2.

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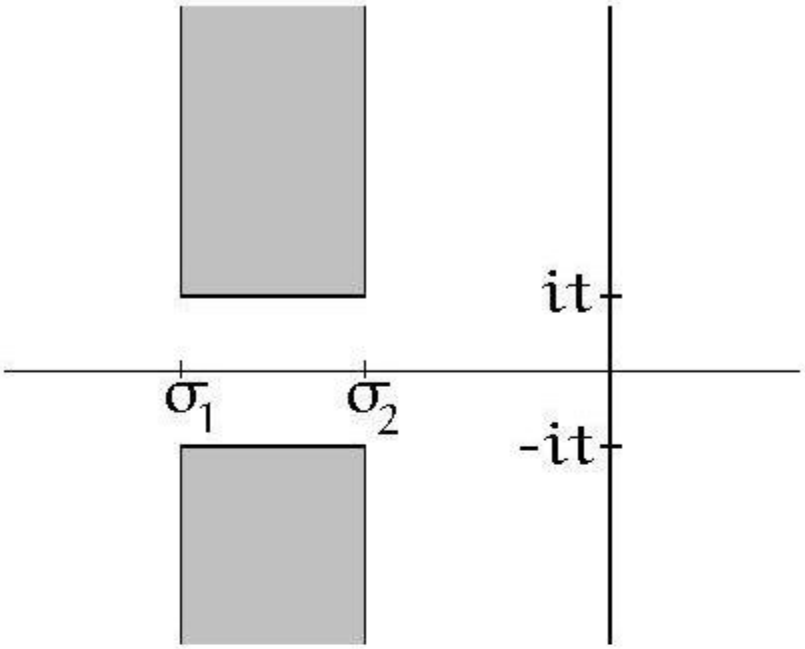

Figure 1

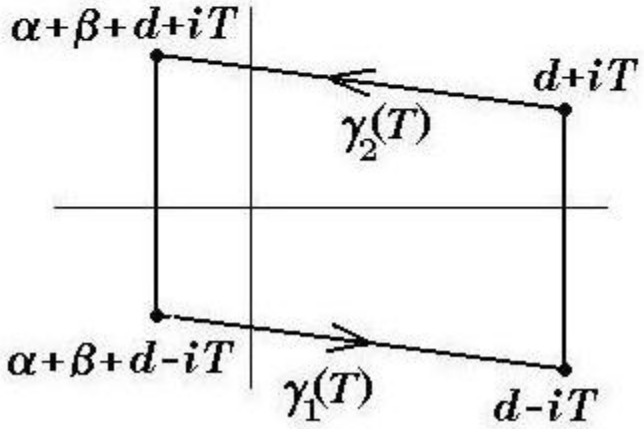

Figure 2